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Conjugate stresses of the Seth–Hill strain tensors

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Abstract

In this paper, conjugate stresses of the Seth–Hill strain measures are treated by means of the definition of energy conjugacy and Hill's principal axis method. This approach results in relations between components of two different conjugate stress tensors. The results are valid for distinct as well as coalescent principal stretches. Illustrative examples are solved to demonstrate the simplicity and usefulness of the derived relations between conjugate stress tensors. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The concept of energy conjugacy for stress and strain measures was first introduced by Hill (1968). This concept plays an important role when writing the internal power of a deforming body. Also the virtual work, as the weak form of equilibrium equations, can be established in terms of a stress measure and the variation of its conjugate strain, as a basis for linear and nonlinear analysis of solids and structures.

A stress measure $\mathbf{T}$ is said to be conjugate to a strain measure $\mathbf{E}$ if $\mathbf{T}:\dot{\mathbf{E}}$ represents power per unit reference volume, $\dot{w}$. That is

$$\dot{w} = III\sigma:\mathbf{D} = \mathbf{T}:\dot{\mathbf{E}} \tag{1}$$

where σ and $\mathbf{D}$ are the Cauchy stress and stretching tensors, respectively, and $III = \det(\mathbf{U})$ is the third invariant of the right stretch tensor $\mathbf{U}$. By the spectral decomposition theorem $\mathbf{U}$ can be recast as

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$$\mathbf{U} = \sum_i \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i \tag{2}$$

where $\{\lambda_i\}$ and $\{\mathbf{N}_i\}$ are the principal stretches and corresponding orthonormal eigenvectors of $\mathbf{U}$, respectively.

According to Guo and Man (1992), the Seth–Hill class of strain measure tensors $\mathbf{E}^{(m)}$ indexed by superscript m are defined as:

$$\mathbf{E}^{(m)} = \frac{1}{m}\sum_i \left(\lambda_i^m - 1\right)\mathbf{N}_i \otimes \mathbf{N}_i = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}); \quad \text{if } m \neq 0$$

$$\mathbf{E}^{(0)} = \sum_i \ln(\lambda_i)\mathbf{N}_i \otimes \mathbf{N}_i = \ln \mathbf{U} \tag{3}$$

where $\mathbf{I}$ is the identity tensor. They proposed an explicit formulation for conjugate stress $\mathbf{T}^{(m)}$ for $|m| \geq 3$ considering distinct and coalescent principal stretches. As an example they obtained an explicit representation of $\mathbf{T}^{(-3)}$ in terms of $\mathbf{U}$, $\mathbf{T}^{(-1)}$ and their invariants. Also, the stress measure conjugate to logarithmic strain tensor $\ln \mathbf{U}$, which is a well known strain measure in plasticity, was first derived by Hoger (1987). Following Guo and Man (1992), differentiating Eq. (3), the material time derivative of $\mathbf{E}^{(m)}$ and $\mathbf{E}^{(-m)}$ $(m > 0)$ are obtained as:

$$\dot{\mathbf{E}}^{(m)} = \frac{1}{m}\sum_{r=1}^{m} \mathbf{U}^{m-r}\dot{\mathbf{U}}U^{r-1} \tag{4a}$$

$$\dot{\mathbf{E}}^{(-m)} = \frac{-1}{m}\sum_{r=1}^{m} \mathbf{U}^{r-m}\overline{\mathbf{U}^{-1}}^{\cdot}\mathbf{U}^{1-r} \tag{4b}$$

where $(\dot{})$ represents the material time derivative. Substituting (4a) into identity

$$\mathbf{T}^{(1)}:\dot{\mathbf{E}}^{(1)} = \mathbf{T}^{(m)}:\dot{\mathbf{E}}^{(m)} \tag{5}$$

yields

$$\mathbf{T}^{(1)}:\dot{\mathbf{U}} = \frac{1}{m}\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{T}^{(m)}\mathbf{U}^{r-1}:\dot{\mathbf{U}} \tag{6}$$

The arbitrariness of $\dot{\mathbf{U}}$ implies that the stress measure tensor $\mathbf{T}^{(m)}$ conjugate to $\mathbf{E}^{(m)}$ satisfies the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = m\mathbf{T}^{(1)} \tag{7}$$

Analogously, for negative integers $-m (m > 0)$, $\mathbf{T}^{(-m)}$ conjugate to $\mathbf{E}^{(-m)}$ satisfies the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = m\mathbf{U}^{m-1}\mathbf{T}^{(-1)}\mathbf{U}^{m-1} \tag{8}$$

which differs from Eq. (7) only by the right-hand side. Hence, Guo and Man (1992) concluded that the crucial point in obtaining an explicit formula for the conjugate stress $\mathbf{T}^{(m)}$ or $\mathbf{T}^{(-m)}$ ($m > 0$) is to solve the tensor equation

$$\sum_{r=1}^{m} \mathbf{U}^{m-r}\mathbf{X}\mathbf{U}^{r-1} = \mathbf{C} \tag{9}$$

where $\mathbf{C}$ is a given symmetric tensor.

In this paper, utilizing the Hill's energy conjugacy notion and principal axis method, some useful relations between different stress measures conjugate to the Seth–Hill strain tensors are derived for $m \neq 0$. The relations for conjugate stresses are not derived through the solution of Eq. (9) as Guo and Man (1992) did. The results are valid for distinct and coalescent principal stretches. Finally, some illustrative examples are solved to show the simplicity and usefulness of the derived relations between conjugate stress tensors.

2. Preliminaries

Let $\mathbf{F}$ denote the deformation gradient at a point of a deforming body. Since $\det(\mathbf{F}) > 0$, the polar decomposition theorem states that $\mathbf{F}$ is uniquely decomposed as

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{10}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively. $\mathbf{U}$ and $\mathbf{V}$ are positive definite symmetric tensors and $\mathbf{R}$ is a proper orthogonal tensor.

The eigenvalues of $\mathbf{U}$ and $\mathbf{V}$, called principal stretches, are denoted by λ_1, λ_2 and λ_3. The principal invariants of $\mathbf{U}$ and $\mathbf{V}$ are

$$I = \lambda_1 + \lambda_2 + \lambda_3$$

$$II = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$III = \lambda_1\lambda_2\lambda_3 \tag{11}$$

According to Cayley–Hamilton theorem, every tensor satisfies its own characteristic equation. That is, for the second order tensor $\mathbf{U}$

$$\mathbf{U}^3 - I\,\mathbf{U}^2 + II\,\mathbf{U} - III\,\mathbf{I} = 0 \tag{12}$$

Some of the well known strain measures as special cases of Eq. (3) and their conjugate stresses are as follows (Hill, 1978; Guo and Dubey, 1984):

(i) Green's strain tensor $\mathbf{E}^{(2)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$. The second Piola–Kirchhoff stress tensor;

$$\mathbf{T}^{(2)} = III\,\mathbf{F}^{-1}\sigma\mathbf{F}^{-T}. \tag{13a}$$

(ii) Almansi strain tensor $\mathbf{E}^{(-2)} = \frac{1}{2}(\mathbf{I} - \mathbf{U}^{-2})$. The weighted convected stress tensor;

$$\mathbf{T}^{(-2)} = III\,\mathbf{F}^{\mathrm{T}}\sigma\mathbf{F}. \tag{13b}$$

(iii) Nominal strain tensor $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$. The Jaumann stress tensor;

$$\mathbf{T}^{(1)} = \frac{1}{2}\left(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}\right). \tag{13c}$$

$\mathbf{E}^{(1)}$ and $\mathbf{T}^{(1)}$ are also called the Biot strain and stress tensors, respectively (Ogden, 1984).
(iv) Logarithmic strain tensor $\mathbf{E}^{(0)} = \ln(\mathbf{U})$. The stress measure $\mathbf{T}^{(0)}$ conjugate to $\ln(\mathbf{U})$ found by Hoger (1987).

3. The stress measure conjugate to $E^{(n)}$

In this section a relation is derived to determine the stress measure $\mathbf{T}^{(n)}$ conjugate to the strain measure $\mathbf{E}^{(n)}$ defined by Eq. (3), in terms of another known stress measure $\mathbf{T}^{(m)}$ (m and n are integers). Consider the identity

$$\mathbf{T}^{(n)}:\dot{\mathbf{E}}^{(n)} = \mathbf{T}^{(m)}:\dot{\mathbf{E}}^{(m)} \tag{14}$$

Using (4a) and (6), the above equation results in:

$$\frac{1}{n}\left(\sum_{r=1}^{n}\mathbf{U}^{n-r}\mathbf{T}^{(n)}\mathbf{U}^{r-1}\right):\dot{\mathbf{U}} = \frac{1}{m}\left(\sum_{r=1}^{m}\mathbf{U}^{m-r}\mathbf{T}^{(m)}\mathbf{U}^{r-1}\right):\dot{\mathbf{U}} \tag{15}$$

Since $\mathbf{U}$ is arbitrary, it is concluded that

$$\frac{1}{n}\sum_{r=1}^{n}\mathbf{U}^{n-r}\mathbf{T}^{(n)}\mathbf{U}^{r-1} = \frac{1}{m}\sum_{r=1}^{m}\mathbf{U}^{m-r}\mathbf{T}^{(m)}\mathbf{U}^{r-1} \tag{16}$$

Using Hill's principal axis method and decomposing the tensors $\mathbf{T}^{(n)}$ and $\mathbf{T}^{(m)}$ under the principal frame $\{\mathbf{N}_i\}$ of $\mathbf{U}$ result in:

$$\mathbf{T}^{(n)} = \sum_{i,j}\mathbf{T}_{ij}^{(n)}\mathbf{N}_i \otimes \mathbf{N}_j \tag{17a}$$

$$\mathbf{T}^{(m)} = \sum_{i,j}\mathbf{T}_{ij}^{(m)}\mathbf{N}_i \otimes \mathbf{N}_j \tag{17b}$$

Substituting (17) into (16), using $\mathbf{U}^n\mathbf{N}_i = \lambda_i^n\mathbf{N}_i$, we arrive at

$$\frac{1}{n}\sum_{r=1}^{n}\lambda_i^{n-r}\mathbf{T}_{ij}^{(n)}\lambda_j^{r-1} = \frac{1}{m}\sum_{r=1}^{m}\lambda_i^{m-r}\mathbf{T}_{ij}^{(m)}\lambda_j^{r-1} \tag{18}$$

This leads to the following useful formula

$$\mathbf{T}_{ij}^{(n)} = \frac{n}{m}T_{ij}^{(m)}\frac{\sum_{r=1}^{m}\lambda_i^{m-r}\lambda_j^{r-1}}{\sum_{r=1}^{n}\lambda_i^{n-r}\lambda_j^{r-1}} \tag{19}$$

Eq. (19) may be used to calculate $\mathbf{T}_{ij}^{(n)}$ in terms of $\mathbf{T}_{ij}^{(m)}$ for distinct and coalescent principal stretches. However, if the principal stretches are distinct, multiplying the enumerator and denominator of Eq. (19) by $(\lambda_i - \lambda_j)$ we obtain:

$$\mathbf{T}_{ij}^{(n)} = \frac{n}{m}\mathbf{T}_{ij}^{(m)}\frac{\lambda_i^m - \lambda_j^m}{\lambda_i^n - \lambda_j^n}; \quad i \neq j \tag{20a}$$

$$\mathbf{T}_{ii}^{(n)} = \mathbf{T}_{ii}^{(m)}\lambda_i^{m-n} \tag{20b}$$

Eq. (20) may be used to obtain the off-diagonal and diagonal members of $\mathbf{T}_{ij}^{(n)}$, respectively.

An analogous procedure is performed to find the stress measure $\mathbf{T}^{(-n)}$ conjugate to the strain measure $\mathbf{E}^{(-n)}$ defined by Eq. (3), in terms of another known stress measure $\mathbf{T}^{(-m)}$ (m and n are positive integers). Writing the identity

$$\mathbf{T}^{(-n)}:\dot{\mathbf{E}}^{(-n)} = \mathbf{T}^{(-m)}:\dot{\mathbf{E}}^{(-m)} \tag{21}$$

and using Eq. (4) gives rise to

$$\mathbf{T}^{(n)}:\left(\frac{1}{n}\sum_{r=1}^{n}\mathbf{U}^{n-r}\dot{\mathbf{U}}\mathbf{U}^{r-1}\right) = \mathbf{T}^{(-m)}:\left(\frac{-1}{m}\sum_{r=1}^{m}\mathbf{U}^{r-m}\dot{\overline{\mathbf{U}^{-1}}}\mathbf{U}^{1-r}\right) \tag{22}$$

The arbitrariness of overline $\dot{\overline{\mathbf{U}^{-1}}}$ and use of Eq. (17) for $\mathbf{T}^{(-m)}$ and $\mathbf{T}^{(-n)}$ leads to the following equation:

$$\mathbf{T}_{ij}^{(-n)} = \frac{n}{m}\mathbf{T}_{ij}^{(-m)}\frac{\sum_{r=1}^{m}\lambda_i^{r-m}\lambda_j^{1-r}}{\sum_{r=1}^{n}\lambda_i^{r-n}\lambda_j^{1-r}} \tag{23}$$

Analogous to (19), for distinct principal stretches, (23) can be simplified to:

$$\mathbf{T}_{ij}^{(-n)} = \frac{n}{m}\mathbf{T}_{ij}^{(-m)}\frac{\lambda_i^{-m} - \lambda_j^{-m}}{\lambda_i^{-n} - \lambda_j^{-n}}; \quad i \neq j \tag{24a}$$

$$\mathbf{T}_{ii}^{(-n)} = \mathbf{T}_{ii}^{(-m)}\lambda_i^{-m+n} \tag{24b}$$

It is noted that use of $-m$ and $-n$ instead of m and n in Eq. (20) yields the same results. Deriving a similar equation for n and $-m$ needs some more efforts. We have

$$\mathbf{T}^{(n)}:\dot{\mathbf{E}}^{(n)} = \mathbf{T}^{(-m)}:\dot{\mathbf{E}}^{(-m)} \tag{25}$$

From Eq. (4) we get

$$\mathbf{T}^{(n)}:\left(\frac{1}{n}\sum_{r=1}^{n}\mathbf{U}^{n-r}\dot{\mathbf{U}}\mathbf{U}^{r-1}\right) = \mathbf{T}^{(-m)}:\left(\frac{-1}{m}\sum_{r=1}^{m}\mathbf{U}^{r-m}\dot{\overline{\mathbf{U}^{-1}}}\mathbf{U}^{1-r}\right) \tag{26}$$

Trying to find a suitable expression for $\dot{\overline{\mathbf{U}^{-1}}}$, we build up the time derivative of $\mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$. Therefore,

$$\dot{\overline{\mathbf{U}^{-1}}} = -\mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1} \tag{27}$$

Substituting Eq. (27) into (26) results in

$$\mathbf{T}^{(n)}:\left(\frac{1}{n}\sum_{r=1}^{n}\mathbf{U}^{n-r}\dot{\mathbf{U}}\mathbf{U}^{r-1}\right) = \mathbf{T}^{(-m)}:\left(\frac{1}{m}\sum_{r=1}^{m}\mathbf{U}^{r-m-1}\dot{\mathbf{U}}\mathbf{U}^{-r}\right) \tag{28}$$

or equivalently,

$$\left(\frac{1}{n}\sum_{r=1}^{n}\mathbf{U}^{n-r}\mathbf{T}^{(n)}\mathbf{U}^{r-1}\right):\dot{\mathbf{U}} = \left(\frac{1}{m}\sum_{r=1}^{m}\mathbf{U}^{r-m-1}\mathbf{T}^{(-m)}\mathbf{U}^{-r}\right):\dot{\mathbf{U}} \tag{29}$$

From the arbitrariness of $\dot{\mathbf{U}}$ and use of Eqs. (17) and (29) we obtain:

$$\frac{1}{n}\sum_{r=1}^{n}\lambda_i^{n-r}\mathbf{T}_{ij}^{(n)}\lambda_j^{r-1} = \frac{1}{m}\sum_{r=1}^{m}\lambda_i^{r-m-1}\mathbf{T}_{ij}^{(-m)}\lambda_j^{-r} \tag{30}$$

or

$$\mathbf{T}_{ij}^{(n)} = \frac{n}{m}\mathbf{T}_{ij}^{(-m)}\frac{\sum_{r=1}^{m}\lambda_i^{r-m-1}\lambda_j^{-r}}{\sum_{r=1}^{n}\lambda_i^{n-r}\lambda_j^{r-1}} \tag{31}$$

Hence, for distinct principal stretches

$$\mathbf{T}_{ij}^{(n)} = -\frac{n}{m}\mathbf{T}_{ij}^{(-m)}\frac{\lambda_i^{-m}-\lambda_j^{-m}}{\lambda_i^{n}-\lambda_j^{n}}; \quad i\neq j \tag{32a}$$

$$\mathbf{T}_{ii}^{(n)} = \mathbf{T}_{ii}^{(-m)}\lambda_i^{-(m+n)} \tag{32b}$$

which could be obtained from Eq. (19) by substituting $-m$ instead of m.

Therefore, it is concluded that Eq. (20), are the basic equations for writing $\mathbf{T}_{ij}^{(n)}$ in terms of $\mathbf{T}_{ij}^{(m)}$ for positive and negative integers m and n. It is noted that Hill (1978) has introduced general transformation formulas in this aspect, with which the same result can be derived through different mathematical manipulation.

Eq. (20) can be written in the form:

$$\left[\mathbf{T}_{ij}^{(n)}\right] = \frac{n}{m}\begin{bmatrix} \frac{m}{n}\mathbf{T}_{11}^{(m)}\lambda_1^{m-n} & & \text{Sym.} \\ \mathbf{T}_{12}^{(m)}\frac{\lambda_1^m-\lambda_2^m}{\lambda_1^n-\lambda_2^n} & \frac{m}{n}\mathbf{T}_{22}^{(m)}\lambda_2^{m-n} & \\ \mathbf{T}_{13}^{(m)}\frac{\lambda_1^m-\lambda_3^m}{\lambda_1^n-\lambda_3^n} & \mathbf{T}_{23}^{(m)}\frac{\lambda_2^m-\lambda_3^m}{\lambda_2^n-\lambda_3^n} & \frac{m}{n}\mathbf{T}_{33}^{(m)}\lambda_3^{m-n} \end{bmatrix} \tag{33}$$

Multiplying Eq. (30) by $\mathbf{N}_i \otimes \mathbf{N}_j$ and summing over i and j gives

$$\frac{1}{n}\sum_{r=1}^{n}\mathbf{U}^{n-r}\mathbf{T}^{(n)}\mathbf{U}^{r-1}=\frac{1}{m}\sum_{r=1}^{m}\mathbf{U}^{m-r}\mathbf{T}^{(m)}\mathbf{U}^{r-1} \tag{34}$$

Pre- and post-multiplying Eq. (34) by $\mathbf{U}$ and $\mathbf{U}^{-1}$, respectively, and subtracting the result from Eq. (34) results in

$$\frac{1}{n}\left(\mathbf{U}^{n}\mathbf{T}^{(n)}\mathbf{U}^{-1}-\mathbf{T}^{(n)}\mathbf{U}^{n-1}\right)=\frac{1}{m}\left(\mathbf{U}^{m}\mathbf{T}^{(m)}\mathbf{U}^{-1}-\mathbf{T}^{(m)}\mathbf{U}^{m-1}\right)$$

and post-multiplying by $\mathbf{U}$ gives

$$\frac{1}{n}\left(\mathbf{U}^{n}\mathbf{T}^{(n)}-\mathrm{T}^{(n)}\mathbf{U}^{n}\right)=\frac{1}{m}\left(\mathbf{U}^{m}\mathbf{T}^{(m)}-\mathbf{T}^{(m)}\mathbf{U}^{m}\right)=\mathbf{C}_1 \tag{35}$$

According to Eq. (3), we have $\mathbf{U}^n=n\mathbf{E}^{(n)}+\mathbf{I}$. Consequently, from Eq. (35) we obtain:

$$\mathbf{E}^{(n)}\mathbf{T}^{(n)}-\mathbf{T}^{(n)}\mathbf{E}^{(n)}=\mathbf{E}^{(m)}\mathbf{T}^{(m)}-\mathbf{T}^{(m)}\mathbf{E}^{(m)}=\mathbf{C}_2 \tag{36}$$

It is noted that $\mathbf{T}^{(n)}$ can not be obtained from Eq. (35) because $\mathbf{C}_1$ is a skew symmetric tensor. Eqs. (35) and (36) are true for any nonzero integers m and n for both distinct and coalescent principal stretches.

From Eq. (33), the conjugate stress measure $\mathbf{T}_{ij}^{(n)}$ can be expressed in terms or as a special weighted form of $\mathbf{T}_{ij}^{(m)}$. As a special case, for $m=2$, any stress measure $\mathbf{T}^{(n)}$ conjugate to its corresponding member of the Seth–Hill class of strain measures $\mathbf{E}^{(n)}$ can be expressed in terms of the second Piola–Kirchhoff stress tensor, $\mathbf{T}^{(2)}$

The following examples illustrate the application of these equations to find the conjugate stresses.

Example 1. Finding $\mathbf{T}^{(1)}$, the Jaumann stress tensor, in terms of $\mathbf{T}^{(2)}$ the second Piola–Kirchhoff stress tensor. Substituting $n=1$ and $m=2$ into Eq. (20) gives rise to

$$\mathbf{T}_{ij}^{(1)}=\frac{1}{2}\mathbf{T}_{ij}^{(2)}\frac{\lambda_i^2-\lambda_j^2}{\lambda_i-\lambda_j}=\frac{1}{2}\mathbf{T}_{ij}^{(2)}\left(\lambda_i+\lambda_j\right) \tag{37}$$

Multiplying Eq. (37) by $\mathbf{N}_i\otimes\mathbf{N}_j$ and summing over i and j yields:

$$\mathbf{T}^{(1)}=\frac{1}{2}\sum_{i,j}\mathbf{T}_{ij}^{(2)}\left(\lambda_i+\lambda_j\right)\mathbf{N}_i\otimes\mathbf{N}_j \tag{38a}$$

or in a basis-free form

$$\mathbf{T}^{(1)}=\frac{1}{2}\left(\mathbf{T}^{(2)}\mathbf{U}+\mathbf{U}\mathbf{T}^{(2)}\right) \tag{38b}$$

which is the same as (13c).

Example 2. Finding $\mathbf{T}^{(-1)}$ in terms of the weighted convected stress tensor, $\mathbf{T}^{(-2)}$. Substituting $n=-1$ and $m=-2$ into Eq. (20) results in

$$\mathbf{T}_{ij}^{(-1)}=\frac{1}{2}\mathbf{T}_{ij}^{(2)}\frac{\lambda_i^{-2}-\lambda_j^{-2}}{\lambda_i^{-1}-\lambda_j^{-1}}=\frac{1}{2}\mathbf{T}_{ij}^{(-2)}\left(\lambda_i^{-1}+\lambda_j^{-1}\right) \tag{39}$$

Multiplying Eq. (39) by $\mathbf{N}_i\otimes\mathbf{N}_j$ and summing over i and j yields:

$$\mathbf{T}^{(-1)} = \frac{1}{2}\sum_{i,j}\mathbf{T}_{ij}^{(-2)}\left(\lambda_i^{-1} + \lambda_j^{-1}\right)\mathbf{N}_i \otimes \mathbf{N}_j \tag{40a}$$

or in a basis-free form

$$\mathbf{T}^{(-1)} = \frac{1}{2}\left(\mathbf{T}^{(-2)}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{T}^{(-2)}\right) \tag{40b}$$

Eq. (40b) represents the stress conjugate to $\mathbf{E}^{(-1)}$ which is the same relation for $\mathbf{T}^{(-1)}$ as cited by Guo and Man (1992).

Example 3. Finding the relation between $\mathbf{T}^{(n)}$ and $\mathbf{T}^{(-n)}$. From Eq. (20) we have $\mathbf{T}_{ij}^{(n)}(\lambda_i^n - \lambda_j^n) = -\mathbf{T}_{ij}^{(-n)}(\lambda_i^{-n} - \lambda_j^{-n})$. Multiplying both sides by $\lambda_i^n\lambda_j^n$ we get

$$\mathbf{T}_{ij}^{(-n)} = \lambda_i^n\lambda_j^n\mathbf{T}_{ij}^{(n)} \tag{41}$$

Multiplying Eq. (41) by $\mathbf{N}_i \otimes \mathbf{N}_j$ and summing over i and j yields:

$$\mathbf{T}^{(-n)} = \mathbf{U}^n\mathbf{T}^{(n)}\mathbf{U}^n \tag{42a}$$

or

$$\mathbf{U}^{-n}\mathbf{T}^{(-n)} = \mathbf{T}^{(n)}\mathbf{U}^n \tag{42b}$$

Therefore, it is enough to find the conjugate stresses only for positive integer n and then, from Eq. (42), find the corresponding stresses for $-n$. It is noted that using the Cayley–Hamilton theorem, the order of $\mathbf{U}$ in Eq. (42) may be reduced.

From Eq. (3), we have $\mathbf{U}^n = n\mathbf{E}^{(n)} + \mathbf{I}$. Therefore, from Eq. (42b), the relation between two strain tensors indexed by n and $-n$, and their conjugate stresses is established as

$$\left(-n\mathbf{E}^{(-n)} + \mathbf{I}\right)\mathbf{T}^{(-n)} = \mathbf{T}^{(n)}\left(n\mathbf{E}^{(n)} + \mathbf{I}\right) \tag{43}$$

Example 4. An alternative method to obtain $\mathbf{T}^{(-1)}$. From Eqs. (38) and (42a) we have

$$\mathbf{T}^{(-1)} = \mathbf{U}\mathbf{T}^{(1)}\mathbf{U} = \frac{1}{2}\mathbf{U}\left(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}\right)\mathbf{U} \tag{44a}$$

$$\mathbf{T}^{(2)} = \mathbf{U}^{-2}\mathbf{T}^{(-2)}\mathbf{U}^{-2} \tag{44b}$$

Substituting Eq. (44b) into (44a) results in

$$\mathbf{T}^{(-1)} = \frac{1}{2}\left(\mathbf{T}^{(-2)}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{T}^{(-2)}\right)$$

which is the same as Eq. (40).

4. Conclusions

For positive and negative integers m and n, relations between the components of $\mathbf{T}^{(m)}$ and $\mathbf{T}^{(n)}$ under the principal frame of $\mathbf{U}$ are derived using Hill's energy conjugacy notion together with Hill's principal axis method. These relations result in tensor equations between $\mathbf{T}^{(m)}$ and $\mathbf{T}^{(n)}$. In consequence, it is shown that $\mathbf{T}^{(-n)}$ can be expressed in terms of $\mathbf{T}^{(n)}$ and powers of $\mathbf{U}$, which can be reduced to two by making use of the Caley–Hamilton theorem. Thus, there is no need to find $\mathbf{T}^{(n)}$ for both positive and negative n.

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